

FREE LOCALLY CONVEX SPACES WITH A SMALL BASE

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ABSTRACT. The paper studies the free locally convex space $L(X)$ over a Tychonoff space X . Since for infinite X the space $L(X)$ is never metrizable (even not Fréchet-Urysohn), a possible applicable generalized metric property for $L(X)$ is welcome. We propose a concept (essentially weaker than first-countability) which is known under the name a \mathfrak{G} -base. A space X has a \mathfrak{G} -base if for every $x \in X$ there is a base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighborhoods at x such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, where $\alpha = (\alpha(n))_{n \in \mathbb{N}} \leq \beta = (\beta(n))_{n \in \mathbb{N}}$ if $\alpha(n) \leq \beta(n)$ for all $n \in \mathbb{N}$. We show that if X is an Ascoli σ -compact space, then $L(X)$ has a \mathfrak{G} -base if and only if X admits an Ascoli uniformity \mathcal{U} with a \mathfrak{G} -base. We prove that if X is a σ -compact Ascoli space of $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type, then $L(X)$ has a \mathfrak{G} -base. As an application we show: (1) if X is a metrizable space, then $L(X)$ has a \mathfrak{G} -base if and only if X is σ -compact, and (2) if X is a countable Ascoli space, then $L(X)$ has a \mathfrak{G} -base if and only if X has a \mathfrak{G} -base.

1. INTRODUCTION

The class of free locally convex spaces $L(X)$ over a (Tychonoff) space X is one of the most important classes in the category of locally convex spaces and continuous operators. This class was introduced by Markov [22] and intensively studied over the last half-century, see for example [1, 12, 15, 25, 27]. Recall that the *free locally convex space* $L(X)$ over a space X is a pair consisting of a locally convex space $L(X)$ and a continuous mapping $i : X \rightarrow L(X)$ such that every continuous mapping f from X to a locally convex space E gives rise to a unique continuous linear operator $\bar{f} : L(X) \rightarrow E$ with $f = \bar{f} \circ i$. The free locally convex space $L(X)$ always exists and is unique.

It is well-known that $L(X)$ is metrizable if and only if X is finite. Moreover, $L(X)$ is a k -space if and only if X is a countable discrete space, see [14]. Therefore, seeking for concrete objects $L(X)$ carrying some *small base* at zero might be interesting for specialist both from topology and functional analysis.

One of such possible concepts extending metrizability is related with locally convex spaces having a \mathfrak{G} -base. Following [19], a topological space X has a \mathfrak{G} -base at a point $x \in X$ if it has a base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of neighborhoods at x such that $U_\beta \subseteq U_\alpha$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$, where $\alpha = (\alpha(n))_{n \in \mathbb{N}} \leq \beta = (\beta(n))_{n \in \mathbb{N}}$ if $\alpha(n) \leq \beta(n)$ for all $n \in \mathbb{N}$; X has a \mathfrak{G} -base if it has a \mathfrak{G} -base at each point $x \in X$.

Originally, the concept of a \mathfrak{G} -base has been formally introduced in [11] in the realm of locally convex spaces for studying (DF) -spaces, $C(X)$ -spaces and spaces in the class \mathfrak{G} in the sense of Cascales and Orihuela, see [20]. Every quasibarrelled locally convex space with a \mathfrak{G} -base has countable tightness both in the original and the weak topology, respectively; each precompact set in a locally convex space with a \mathfrak{G} -base is metrizable, see again [20]. It is easy to see that every metrizable group has a \mathfrak{G} -base at the identity. Topological groups with a \mathfrak{G} -base are thoroughly studied in [19], see also [4, 16, 18].

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Being motivated by several results of the above type (see [20] also for a long list of references), the authors in [19] posed the following general problem:

Problem 1.1 ([19]). *For which spaces X the free locally convex space $L(X)$ has a \mathfrak{G} -base?*

For a space X we denote by $C_p(X)$ and $C_k(X)$ the space $C(X)$ of all continuous real-valued functions on X endowed with the pointwise topology τ_p and the compact-open topology τ_k , respectively. Recall that a space X is called an *Ascoli space* if every compact subset \mathcal{K} of $C_k(X)$ is evenly continuous [2]. In other words, X is Ascoli if and only if the compact-open topology of $C_k(X)$ is Ascoli in the sense of [23, p.45].

Using a deep result of Uspenskiĭ [27], for a wide class of topological spaces X we show that Problem 1.1 can be reformulated in the term of function spaces $C(X)$.

Theorem 1.2. *Let X be a Dieudonné complete Ascoli space (in particular, X is a paracompact k -space or a metrizable space). Then $L(X)$ has a \mathfrak{G} -base if and only if $C_k(X)$ has a compact resolution swallowing compact subsets.*

Recall that a family $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact subsets of a space Z is called a *compact resolution* if \mathcal{K} covers Z and $K_\alpha \subseteq K_\beta$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^\mathbb{N}$. Following Christensen [5], we say that \mathcal{K} *swallows compact sets* of Z if for every compact subset K of Z there is an $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq K_\alpha$. The importance of this concept follows from the following deep result of Christensen: *A metrizable and separable space Z is Polish if and only if Z has a compact resolution swallowing compact sets.* Consequently, since $C_k(X)$ is Polish if X is locally compact metrizable and separable, by Theorem 1.2 the space $L(X)$ has a \mathfrak{G} -base. These results and Theorem 1.2 motivate the following question:

Problem 1.3. *For which spaces X , the space $C_k(X)$ has a compact resolution (swallowing compact sets)?*

This problem is of independent interest because (see for example [20, Theorem 9.9]) $C_k(X)$ has a compact resolution if and only if $C_k(X)$ is K -analytic, i.e. $C_k(X)$ is the image under an upper semi-continuous compact-valued map defined in $\mathbb{N}^\mathbb{N}$; the same result holds for $C_p(X)$, see [26]. Moreover, Tkachuk proved in [26] that $C_p(X)$ has a compact resolution swallowing compact sets if and only if X is a countable discrete space.

Christensen had already proved the following result (see also Corollary 2.3 below): *If X is a separable metrizable space, then $C_k(X)$ has a compact resolution if and only if X is σ -compact.* Below we strengthen this result by showing that under the same assumption on X the space $C_k(X)$ has even a compact resolution swallowing compact sets, see Corollary 2.10 below. These results motivate the question: *For which σ -compact spaces X the space $C_k(X)$ has a compact resolution (swallowing compact sets)?* The aforementioned results explain our study of functions spaces with compact resolutions, see Section 2.

In Section 3 we prove Theorem 1.2 and obtain the following partial answers to Problem 1.1.

Theorem 1.4. *Let X be an Ascoli σ -compact space. Then $L(X)$ has a \mathfrak{G} -base if and only if X admits an Ascoli uniformity \mathcal{U} with a \mathfrak{G} -base.*

Theorem 1.4 needs a new concept which is stronger than to be an Ascoli space.

Definition 1.5. A uniformity \mathcal{U} on a space X is said to be *Ascoli* if \mathcal{U} is admissible and any compact subset K of $C_k(X)$ is uniformly equicontinuous with respect to \mathcal{U} , i.e. for every $\varepsilon > 0$ there is $U \in \mathcal{U}$ such that $|f(x) - f(y)| < \varepsilon$ for every $f \in K$ and each $(x, y) \in U$. We say that X is a *uniformly Ascoli space* if X has an Ascoli uniformity.

We provide also a sufficient condition on a σ -compact space X for which $L(X)$ has a \mathfrak{G} -base. This approach requires some additional concept.

Definition 1.6. A topological space X is a space of $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type if for every compact subset K of X the set $\Delta_K = \{(x, x) \in X \times X : x \in K\}$ has an $\mathbb{N}^{\mathbb{N}}$ -decreasing base $\{U_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ of open neighborhoods in $X \times X$, i.e. for every open neighborhood U of Δ_K there is $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that $\Delta_K \subseteq U_\alpha \subseteq U$.

Theorem 1.7. *Let X be a σ -compact Ascoli space. If X is of $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type, then $L(X)$ has a \mathfrak{G} -base.*

It is easy to show, see Proposition 2.7 below, that if X is a metrizable space or a countable space such that every point $x \in X$ has a \mathfrak{G} -base, then X is of $\mathbb{N}^{\mathbb{N}}$ -uniformly compact type. So Theorem 1.7 with Corollary 2.10 imply

Corollary 1.8. *If X is a metrizable space, then $L(X)$ has a \mathfrak{G} -base if and only if X is σ -compact.*

In particular, the space $L(\mathbb{Q})$ has a \mathfrak{G} -base.

Corollary 1.9. *If X is a countable Ascoli space, then $L(X)$ has a \mathfrak{G} -base if and only if X has a \mathfrak{G} -base.*

Note that Corollaries 1.8 and 1.9 are proved independently in [3] using different methods.

2. COMPACT RESOLUTIONS IN FUNCTION SPACES

Recall that a subset A of a topological space X is called *functionally bounded* if every continuous function $f \in C(X)$ is bounded on A . Recall also that a *resolution* $\mathcal{A} = \{A_\alpha : \alpha \in \mathbb{N}^{\mathbb{N}}\}$ in X is a cover of X such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$ for all $\alpha, \beta \in \mathbb{N}^{\mathbb{N}}$. If all A_α are functionally bounded, the resolution \mathcal{A} is called *functionally bounded*. Note also that by a result of Calbrix, see [20, Theorem 9.7], if $C_p(X)$ is analytic, then X is σ -compact. Recall that a space Z is called *analytic* if it is a continuous image of $\mathbb{N}^{\mathbb{N}}$.

We shall use the following fact, see Corollary 9.1 of [20]. Recall that a (Tychonoff) space X is *cosmic* if it is a continuous image of a separable metric space.

Fact 2.1. *Let X be a cosmic space. Then $C_p(X)$ has a functionally bounded resolution if and only if X is σ -compact. Consequently, if $C_k(X)$ has a compact resolution, then X is σ -compact.*

Proposition 2.2. *Let X be a paracompact first countable space such that $C_k(X)$ is an angelic space. Then X is Lindelöf.*

Proof. If X is not Lindelöf, the space $C_k(X)$ contains a closed subset A homeomorphic to \mathbb{N}^{ω_1} by Lemma 1 of [24]. But the space \mathbb{N}^{ω_1} is not angelic, so $C_k(X)$ is also not angelic. This contradiction shows that X must be Lindelöf. \square

This yields the following

Corollary 2.3. *Let X be a metrizable space. If $C_k(X)$ has a functionally bounded resolution, then X is σ -compact.*

Proof. Proposition 9.6 of [20] implies that $C_p(X)$ is angelic. By [13, Theorem, page 31], the space $C_k(X)$ is also angelic. Now Proposition 2.2 implies that X is Lindelöf. So being metrizable, the space X is separable, and hence X is a cosmic space. Thus X is σ -compact by Fact 2.1. \square

The next proposition completes Proposition 2.2.

Proposition 2.4. *Let X be a paracompact Čech-complete space. Then $C_k(X)$ is an angelic space if and only if X is Lindelöf.*

Proof. Assume that $C_k(X)$ is angelic. By a result of Frolík [6, 5.5.9], there is a perfect map f from X onto a complete metrizable space Y . Suppose that X is not Lindelöf. Then, since f is perfect, Y is also not Lindelöf by [6, Theorem 3.8.9]. As $f^{-1}(K)$ is compact for every compact set $K \subseteq Y$ by [6, Theorem 3.7.2], the space $C_k(Y)$ embeds into $C_k(X)$, and hence $C_k(Y)$ is also angelic. Now Proposition 2.2 implies that Y is Lindelöf, a contradiction. Thus X is a Lindelöf space.

Conversely, let X be Lindelöf. Then X has a compact resolution swallowing compact sets, see the proof of Proposition 4.7 in [17]. So $C_p(X)$ is angelic by Example 4.1 and Theorem 4.5 of [20]. Therefore $C_k(X)$ is angelic by [13, Theorem, page 31]. \square

Recall that, for a space X , the family of sets of the form

$$[K, \varepsilon] := \{f \in C(X) : f(K) \subset (-\varepsilon, \varepsilon)\}$$

where K is a compact subset of X , is a base of the compact-open topology τ_k on $C(X)$. Denote by $\delta : X \rightarrow C_k(C_k(X))$ the canonical map defined by

$$\delta(x)(f) := f(x), \quad \forall x \in X, \forall f \in C(X).$$

Proposition 2.5. *Let X be an Ascoli space. If $C_k(X)$ has a compact resolution swallowing compact sets, then X has an Ascoli uniformity \mathcal{U} with a \mathfrak{G} -base.*

Proof. Let $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a compact resolution swallowing compact sets of $C_k(X)$. Then, by [9], the space $C_k(C_k(X))$ has a \mathfrak{G} -base $\{V_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$, where $V_\alpha := [K_\alpha, \alpha(1)^{-1}]$. Since X is Ascoli space, the canonical map $\delta : X \rightarrow C_k(C_k(X))$ is an embedding by Corollary 5.8 of [2]. For every $\alpha \in \mathbb{N}^\mathbb{N}$, define

$$U_\alpha := \{(x, y) \in X \times X : \delta(x) - \delta(y) \in V_\alpha\},$$

and let \mathcal{U} be the uniformity on X induced from $C_k(C_k(X))$. Clearly, $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a \mathfrak{G} -base for \mathcal{U} and \mathcal{U} is admissible. We show that \mathcal{U} is also Ascoli.

Fix a compact subset K of $C_k(X)$ and $\varepsilon > 0$. Take $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq K_\alpha$ and $\alpha(1) > 1/\varepsilon$. Now for every $(x, y) \in U_\alpha$ and each $f \in K$, we obtain

$$|f(x) - f(y)| = |(\delta(x) - \delta(y))(f)| < \frac{1}{\alpha(1)} < \varepsilon,$$

Thus \mathcal{U} is an Ascoli uniformity. \square

We shall use the following encoding operation of elements of $\mathbb{N}^\mathbb{N}$. We encode each $\alpha \in \mathbb{N}^\mathbb{N}$ into a sequence $\{\alpha_i\}_{i \in \omega}$ of elements of $\mathbb{N}^\mathbb{N}$ as follows. Consider an arbitrary decomposition of \mathbb{N} onto a disjoint family $\{N_i\}_{i \in \omega}$ of infinite sets, where $N_i = \{n_{k,i}\}_{k \in \mathbb{N}}$. Now for $\alpha = (\alpha(n))_{n \in \mathbb{N}}$ and $i \in \omega$, we set $\alpha_i = (\alpha_i(k))_{k \in \mathbb{N}}$, where $\alpha_i(k) := \alpha(n_{k,i})$ for every $k \in \mathbb{N}$. Conversely, for every sequence $\{\alpha_i\}_{i \in \omega}$ of elements of $\mathbb{N}^\mathbb{N}$ we define $\alpha = (\alpha(n))_{n \in \mathbb{N}}$ setting $\alpha(n) := \alpha_i(k)$ if $n = n_{k,i}$.

For a subset A of a set S , a subset B of $S \times S$ and $(a, b) \in B$, we define

$$\Delta_A := \{(a, a) \in S \times S : a \in A\} \text{ and } B(a) := \{s \in S : (a, s) \in B\}.$$

Next definition generalizes the classical notion of spaces of pointwise countable type (due to Arhangel'skii) and also Definition 1.6.

Definition 2.6. Let I be an ordered set and X a topological space. The space X is a space of

- (i) *I-compact type* if every compact subset K of X has an decreasing I -base $\{U_i : i \in I\}$ of open neighborhoods, i.e. $U_i \subseteq U_j$ for all $i \geq j$ and for every open neighborhood U of K there is $i \in I$ such that $K \subseteq U_i \subseteq U$;
- (ii) *I-pointwise countable type* if for every x in X there exists a compact set K which has a decreasing I -base of open sets;

- (iii) *I-uniformly compact type* if for every compact subset K of X the set Δ_K has an I -decreasing base $\{U_i : i \in I\}$ of open neighborhoods in $X \times X$, i.e. for every open neighborhood U of Δ_K there is $i \in I$ such that $\Delta_K \subseteq U_i \subseteq U$.

As usual a decreasing $\mathbb{N}^\mathbb{N}$ -base of a subset A of X is called a \mathfrak{G} -base of A . Next proposition provides possible two cases when X is of $\mathbb{N}^\mathbb{N}$ -uniformly compact type.

Proposition 2.7. *A Tychonoff space X is of $\mathbb{N}^\mathbb{N}$ -uniformly compact type if one of the following conditions holds:*

- (i) *X is a metrizable space;*
- (ii) *X is a countable space such that every point $x \in X$ has a \mathfrak{G} -base.*

Proof. (i) For every compact subset K of X , the compact subset Δ_K of the metrizable space $X \times X$ has a decreasing base $\{V_n : n \in \mathbb{N}\}$. Then the family $\{U_\alpha = V_{\alpha(1)} : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a \mathfrak{G} -base of K .

(ii) Let $\{x_n : n \in \mathbb{N}\}$ be an enumeration of X and $\{U_{\alpha, x_n} : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a \mathfrak{G} -base at x_n . Fix a compact subset K of X . If K is finite, then clearly the family

$$\left\{ \bigcup_{x_n \in K} U_{\alpha, x_n} \times U_{\alpha, x_n} : \alpha \in \mathbb{N}^\mathbb{N} \right\}$$

is a \mathfrak{G} -base at Δ_K .

Assume that K is infinite. For every $\alpha \in \mathbb{N}^\mathbb{N}$ with the encoding (α_n) , we define

$$U_\alpha := \bigcup \{U_{\alpha_n, x_n} \times U_{\alpha_n, x_n} : x_n \in K\}.$$

We claim that the family $\mathcal{U} = \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a \mathfrak{G} -base at Δ_K . Indeed, fix an open neighborhood U of Δ_K . For every $x_n \in K$ take $\alpha_n \in \mathbb{N}$ such that $U_{\alpha_n, x_n} \times U_{\alpha_n, x_n} \subseteq U$. Now if $\alpha \in \mathbb{N}^\mathbb{N}$ is built by the sequence (α_n) , we obtain $K \subseteq U_\alpha \subseteq U$. Thus \mathcal{U} is a \mathfrak{G} -base at Δ_K . \square

We shall use the following fact which is proved in the “if” part of the Ascoli theorem [6, 3.4.20].

Fact 2.8. *Let X be a space and A be an evenly continuous (in particular, equicontinuous) pointwise bounded subset of $C_k(X)$. Then the closure \bar{A} of A in τ_k is a compact equicontinuous subset of $C_k(X)$.*

If an Ascoli space X is additionally σ -compact, we can reverse Proposition 2.5.

Proposition 2.9. *Let X be a σ -compact space. Then the space $C_k(X)$ has a compact resolution swallowing compact sets if one of the following conditions holds:*

- (i) *X has an Ascoli uniformity \mathcal{U} with a \mathfrak{G} -base;*
- (ii) *X is an Ascoli space of $\mathbb{N}^\mathbb{N}$ -uniformly compact type.*

Proof. Let $X = \bigcup_{n \in \mathbb{N}} C_n$ be the union of an increasing sequence $\{C_n\}_{n \in \mathbb{N}}$ of compact subsets. For the case (i), let $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a \mathfrak{G} -base for the Ascoli uniformity \mathcal{U} . For the case (ii), for every $n \in \mathbb{N}$, let $\{U_{\alpha, n} : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a \mathfrak{G} -base of Δ_{C_n} . For every $\alpha \in \mathbb{N}^\mathbb{N}$ with the encoding $\{\alpha_k\}_{k \in \omega}$, we define

$$A_\alpha := \bigcap_{k \in \mathbb{N}} \{f \in C(X) : |f(x)| \leq \alpha_0(k) \quad \forall x \in C_k\},$$

$$B_\alpha := \bigcap_{n \in \mathbb{N}} \left\{ f \in C(X) : |f(x) - f(y)| \leq \frac{1}{n} \quad \forall (x, y) \in U_{\alpha_n} \right\}, \quad \text{for case (i),}$$

$$B_\alpha := \bigcap_{n \in \mathbb{N}} \left\{ f \in C(X) : |f(x) - f(y)| \leq \frac{1}{n} \quad \forall (x, y) \in U_{\alpha_n, n} \right\}, \quad \text{for case (ii),}$$

and set $K_\alpha := A_\alpha \cap B_\alpha$. Clearly, K_α is closed in the compact-open topology τ_k and $K_\alpha \subseteq K_\beta$ for every $\alpha \leq \beta$. Fix $\alpha \in \mathbb{N}^\mathbb{N}$. By construction, K_α is pointwise bounded. We check that the set K_α is equicontinuous. We distinguish between cases (i) and (ii).

Case (i). Given $\varepsilon > 0$ take $n \in \mathbb{N}$ such that $n > 1/\varepsilon$. Then for every $f \in K_\alpha$, by the definition of B_α , we obtain $|f(x) - f(y)| \leq \frac{1}{n} < \varepsilon$ whenever $(x, y) \in U_{\alpha_n}$. So K_α is equicontinuous.

Case (ii). Fix $x \in X$, so $x \in C_l$ for some $l \in \mathbb{N}$. Given $\varepsilon > 0$ take $n > l$ such that $n > 1/\varepsilon$. Then for every $f \in K_\alpha$, by the definition of B_α , we obtain $|f(x) - f(y)| \leq \frac{1}{n} < \varepsilon$ whenever $(x, y) \in U_{\alpha_n, n}$. Since $U_{\alpha_n, n}(x)$ is an open neighborhood of x , the set K_α is equicontinuous.

Now in both cases (i) and (ii), Fact 2.8 implies that K_α is a compact subset of $C_k(X)$.

Let us show that the family $\mathcal{K} := \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ swallows the compact sets of $C_k(X)$. Fix a compact subset K of $C_k(X)$. Since X is an Ascoli space, K is pointwise bounded and equicontinuous. Define $\alpha_0 = (\alpha_0(k))_{k \in \mathbb{N}}$ as follows: for every $k \in \mathbb{N}$, set

$$\alpha_0(k) := [\sup\{|f(x)| : x \in C_k, f \in K\}] + 1,$$

where $[t]$ is the integral part of a real number t . Again we distinguish between cases (i) and (ii).

Case (i). Since \mathcal{U} is Ascoli, for every $n \in \mathbb{N}$, take $\alpha_n \in \mathbb{N}^\mathbb{N}$ such that $|f(x) - f(y)| \leq 1/n$ for every $f \in K$ and each $(x, y) \in U_{\alpha_n}$. If $\alpha \in \mathbb{N}^\mathbb{N}$ is built by the above procedure we obtain $K \subseteq K_\alpha$.

Case (ii). Fix $n \in \mathbb{N}$. For every $x \in C_n$ take an open neighborhood U_x of x such that $|f(x) - f(y)| \leq 1/2n$ for every $f \in K$ and each $y \in U_x$. Set $W := \bigcup_{x \in C_n} U_x \times U_x$. Then for every $(z, y) \in W$ there is $x \in C_n$ such that $(z, y) \in U_x \times U_x$ and hence

$$|f(z) - f(y)| \leq |f(x) - f(z)| + |f(x) - f(y)| \leq \frac{1}{n}, \text{ for every } f \in K.$$

Since X is of $\mathbb{N}^\mathbb{N}$ -uniformly compact type, we choose $\alpha_n \in \mathbb{N}^\mathbb{N}$ such that $\Delta_{C_n} \subseteq U_{\alpha_n, n} \subseteq W$. If $\alpha \in \mathbb{N}^\mathbb{N}$ is built by the sequence (α_n) , we obtain $K \subseteq K_\alpha$.

Also now in both cases (i) and (ii) the family \mathcal{K} swallows the compact sets of $C_k(X)$. \square

As a corollary we obtain the following strengthening of Christensen's theorem.

Corollary 2.10. *For a metrizable space X , $C_k(X)$ has a compact resolution swallowing the compact sets of $C_k(X)$ if and only if X is σ -compact.*

Proof. If $C_k(X)$ has a compact resolution swallowing the compact sets of $C_k(X)$, then X is σ -compact by Corollary 2.3. The converse assertion follows from Propositions 2.7 and 2.9. \square

In particular, the space $C_k(\mathbb{Q})$ has a compact resolution swallowing its compact sets.

We conclude this section with the following Christensen's type result.

Proposition 2.11. *The following assertions are equivalent.*

- (i) $C_k(X)$ is analytic.
- (ii) $C_k(X)$ is K -analytic and X is σ -compact.
- (iii) $C_k(X)$ has a compact resolution and X is σ -compact.

If additionally X is first countable, the above conditions are equivalent to

- (iv) X is metrizable and σ -compact.

Proof. First we prove the following claim using some ideas from [10] strongly motivated by Fer-rando's Theorem 1 of [7].

Claim. *If X is a σ -compact space, then $C_p(X)$ admits a stronger metrizable locally convex topology. Indeed, let $X = \bigcup_{n=1}^\infty K_n$, where K_n is a compact subset of X and $K_n \subseteq K_{n+1}$ for every $n \in \mathbb{N}$. For every $n \in \mathbb{N}$, define*

$$(2.1) \quad V_n := \left\{ f \in C(X) : \sup_{x \in K_n} |f(x)| \leq \frac{1}{n} \right\}.$$

Clearly, $V_{n+1} \subseteq V_n$ and $\bigcap_{n=1}^{\infty} V_n = \{0\}$, where 0 stands for the identically null function on X . Note that the sets V_n are absorbing since if $g \in C(X)$, then there is $k \in \mathbb{N}$ such that $\sup_{x \in K_n} |g(x)| \leq k$, so that $g \in knV_n$. Moreover, if

$$U = \left\{ f \in C(X) : \max_{1 \leq i \leq n} |f(x_i)| < \epsilon \right\}$$

and $p \in \mathbb{N}$ is chosen so that $x_i \in V_p$ for $1 \leq i \leq n$ and $p^{-1} < \epsilon$, then $V_p \subseteq U$ and clearly $V_{2n} \subseteq 2^{-1}V_n$ for each $n \in \mathbb{N}$. This shows that $\{V_n : n \in \mathbb{N}\}$ is a base of neighborhoods of the origin of a locally convex topology on $C(X)$ stronger than the pointwise topology. The claim is proved.

The implications (i) \Rightarrow (ii) \Rightarrow (iii) are clear; note that if $C_k(X)$ is analytic, then Calbrix's result, see [20, Theorem 9.7], implies that X is σ -compact.

(iii) \Rightarrow (i): If X is σ -compact, the claim and (2.1) show that the space $C(X)$ admits a metrizable locally convex topology weaker (or equal) to the compact-open topology τ_k . As $C_k(X)$ has a compact resolution (by assumption), we apply Talagrand's result, see [20, Proposition 6.3], to deduce that $C_k(X)$ is analytic.

If X first countable, (i) is equivalent to (iv) by [23, Theorem 5.7.5]. \square

3. PROOFS OF THEOREMS 1.2, 1.4 AND 1.7

It is well-known that the dual space of $C_k(X)$ is the space $M_c(X)$ of all regular Borel measures on X with compact support. Denote by τ_e the topology on $M_c(X)$ of uniform convergence on the equicontinuous pointwise bounded subsets of $C(X)$. For $A \subseteq C_k(X)$ and $B \subseteq M_c(X)$, we set as usual

$$A^\circ = \{\mu \in M_c(X) : |\mu(f)| \leq 1 \ \forall f \in A\}, \text{ and } B^\circ = \{f \in C_k(X) : |\mu(f)| \leq 1 \ \forall \mu \in B\}.$$

Proposition 3.1. *Let X be an Ascoli space. Then $(M_c(X), \tau_e)$ has a \mathfrak{G} -base if and only if $C_k(X)$ has a compact resolution swallowing compact subsets of $C_k(X)$.*

Proof. Assume that $C_k(X)$ has a compact resolution swallowing compact subsets of $C_k(X)$. Let $\mathcal{K} = \{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a compact resolution swallowing compact sets of $C_k(X)$. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $U_\alpha := K_\alpha^\circ$. We show that the family $\mathcal{U} := \{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a \mathfrak{G} -base in $(M_c(X), \tau_e)$. Indeed, every U_α is a neighborhood of zero in τ_e because K_α is equicontinuous. Now let U be a neighborhood of zero in $(M_c(X), \tau_e)$. Take an equicontinuous pointwise bounded subset A of $C(X)$ such that $A^\circ \subseteq U$. By Fact 2.8, the closure K of A in $C_k(X)$ is compact. So there is $\alpha \in \mathbb{N}^\mathbb{N}$ such that $K \subseteq K_\alpha$. Clearly, $U_\alpha = K_\alpha^\circ \subseteq A^\circ \subseteq U$. Thus \mathcal{U} is a base of τ_e .

Conversely, let $(M_c(X), \tau_e)$ have a \mathfrak{G} -base $\{U_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$. For every $\alpha \in \mathbb{N}^\mathbb{N}$, set $C_\alpha := U_\alpha^\circ$. We show that the family $\mathcal{C} := \{C_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a compact resolution in $C_k(X)$ swallowing the compact sets.

Clearly, if $\alpha \leq \beta$ then $C_\alpha \subseteq C_\beta$. Since $M_c(X)$ is the dual space of $C_k(X)$, every C_α is closed in $C_k(X)$. To show that C_α is compact in $C_k(X)$, take an absolutely convex neighborhood V of zero in $M_c(X)$ such that $\overline{V} \subseteq U_\alpha$ and choose an equicontinuous pointwise bounded subset A of $C(X)$ such that $A^\circ \subseteq V$. Clearly, the absolutely convex hull $\text{acx}(A)$ of A is also an equicontinuous pointwise bounded subset of $C(X)$. So, by Fact 2.8, the closure $K := \overline{\text{acx}(A)}^{\tau_k}$ of $\text{acx}(A)$ in the compact-open topology τ_k is a compact equicontinuous subset of $C_k(X)$. Since the bounded convex subsets of $C(X)$ in τ_k and $\sigma(C(X), M_c(X))$ are the same, the Bipolar theorem implies that $K = K^{\circ\circ}$. As

$$C_\alpha \subseteq \overline{V}^\circ \subseteq A^{\circ\circ} = K^{\circ\circ} = K$$

we obtain that C_α is compact.

Let C be a compact subset of $C_k(X)$. Since X is Ascoli, C is equicontinuous and clearly pointwise bounded. Take $\alpha \in \mathbb{N}^\mathbb{N}$ such that $U_\alpha \subseteq C^\circ$. Then

$$C \subseteq C^{\circ\circ} \subseteq U_\alpha^\circ = C_\alpha.$$

Thus the family \mathcal{C} swallows the compact sets of $C_k(X)$. \square

For a space X we denote by μX the Dieudonné completion of X . Note that any paracompact space is Dieudonné complete, see [6, 8.5.13(d)]. Now Theorem 1.2 immediately follows from the following more general result.

Theorem 3.2. *Let X be a Tychonoff space such that μX is an Ascoli space. Then $L(X)$ has a \mathfrak{G} -base if and only if $C_k(\mu X)$ has a compact resolution swallowing compact subsets. In this case the space $C_k(\mu X)$ is Lindelöf.*

Proof. The space $L(X)$ has a \mathfrak{G} -base if and only if its (Raikov) completion $\overline{L(X)}$ has a \mathfrak{G} -base, see Proposition 2.7 of [19]. It is known that $\overline{L(X)}$ is $(M_c(\mu X), \tau_e)$, see Theorem 5 of [27]. Now Proposition 3.1 applies. To prove the last assertion we note that the space $C_k(\mu X)$ is K -analytic by Theorem 9.9 of [20]. So $C_k(\mu X)$ is Lindelöf by Proposition 3.13 of [20]. \square

We do not know whether the condition on X to be an Ascoli space is essential in Theorem 1.2.

Question 4.18 in [19] asks whether for a k -space the existence of a \mathfrak{G} -base on $L(X)$ implies that also $C_k(C_k(X))$ has a \mathfrak{G} -base. By Ferrando–Kąkol theorem [10] (see also [19, Theorem 4.9]), the space $C_k(X)$ has a compact resolution swallowing compact subsets if and only if $C_k(C_k(X))$ has a \mathfrak{G} -base. Combining this result with Theorem 1.2 we obtain a partial answer to [19, Question 4.18].

Corollary 3.3. *Let X be a Dieudonné complete Ascoli space. Then $L(X)$ has a \mathfrak{G} -base if and only if the space $C_k(C_k(X))$ has a \mathfrak{G} -base.*

Corollary 1.9 follows from the next result.

Corollary 3.4. *If X is a countable Ascoli space, then the following assertions are equivalent:*

- (i) $L(X)$ has a \mathfrak{G} -base;
- (ii) $C_k(X)$ has a compact resolution swallowing the compact sets of $C_k(X)$;
- (iii) X has a \mathfrak{G} -base.

Proof. (i) \Leftrightarrow (ii) follows from Theorem 1.2 (recall that any countable space being Lindelöf is Dieudonné complete). (i) \Rightarrow (iii) follows from the fact that X is a subspace of $L(X)$. (iii) \Rightarrow (ii) follows from Propositions 2.7 and 2.9. \square

Note that Ferrando in [8] gives a direct proof of the implication (iii) \Rightarrow (ii) in Corollary 3.4.

We provide another necessary condition for a space X to have the space $L(X)$ with a \mathfrak{G} -base.

Proposition 3.5. *If $L(X)$ has a \mathfrak{G} -base, then every precompact set in $L(X)$ (hence also in X) is metrizable.*

Proof. Note that in every locally convex space E with a \mathfrak{G} -base every precompact set is metrizable, see [20, Theorem 11.1]. We conclude the proof by noticing that X embeds into $L(X)$. \square

Recall that a topological space X is a k_ω -space (an \mathcal{MK}_ω -space) if X is the inductive limit of a countable family of compact (compact and metrizable) subsets. We proved in [16] that $L(X)$ has a \mathfrak{G} -base for every \mathcal{MK}_ω -space X . Combining this result with Proposition 3.5 we obtain

Corollary 3.6. *Let X be a k_ω -space. Then $L(X)$ has \mathfrak{G} -base if and only if X is an \mathcal{MK}_ω -space.*

Remark 3.7. In [19, Question 4.19] we ask whether the existence of a \mathfrak{G} -base in the free abelian group $A(X)$ over a space X implies that $L(X)$ has also a \mathfrak{G} -base. Let X be a discrete space. Then it is clear that $A(X)$ being discrete has a \mathfrak{G} -base. In [21] it is shown that if X is of cardinality $\geq \mathfrak{c}$, then $L(X)$ does not have a \mathfrak{G} -base. This answers Question 4.19 of [19] in the negative. Our Corollary 1.8 implies a stronger result: for every uncountable discrete space X , the space $L(X)$ does not have a \mathfrak{G} -base.

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